these constants for various fluids. At the very least, this would be an effective interim method by which elastic non-Newtonian fluids could be classified. On the other hand, an understanding of the underlying mechanism involved could lead to a theoretical basis for α and u_{*cr} which would then make this type of rheometer a basic device. An analysis of this type would be difficult to formulate, since it would involve the determination of the viscoelastic effects on the thickness of the sublayer for a turbulent boundary layer. This would certainly require a procedure for the determination of the sublayer thickness itself which, to date, is not available even for Newtonian fluids. However, with more systematic data available, the entire process will be better understood. It is, therefore, strongly recommended that this type of rheometric device be seriously considered for future classifications of these fluids.

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NOTATION

A = 2.303/k
B = constant in the law-of-the-wall equation
B_N = value of B for Newtonian fluids
ΔB = $B - B_N$
D = I.D. of pipe
f = friction factor, $\tau_w/(\frac{1}{2}\rho\bar{u}^2)$
k = Prandtl mixing length constant equal to 0.4
N_{Re} = Reynolds number, $\bar{u}D/\nu_w$
$\bar{u}$ = mean velocity of flow through the pipe
u_m = maximum flow velocity in the pipe
u_* = friction velocity, $\sqrt{\tau_w/\rho}$
u_{*cr} = critical value of u_*, taken as 0.23 ft./sec. in this report
y = radial distance from the pipe wall

Greek Letters

α = fluid property parameter defined in Equation (3)
ν_w = kinematic viscosity at the wall
ρ = mass density of the fluid
τ_w = shear stress at the wall

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An Experimental Study of Falling Liquid Films

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A new method for measuring wavelength and wave velocity is described, and experimental values for water flowing down a vertical plane are compared with a numerical solution of the Orr-Sommerfeld equation. Good agreement is obtained in the region near the top of the film where small disturbance theory is expected to be valid. Experimental Reynolds numbers ranged from 8 to 120.

There are a wide variety of heat and mass transfer processes which involve transport across a thin liquid film. The current list ranges from ablating space vehicles (*17*) to film cooling of turbine blades (*12*) to the typical chemical engineering problems of gas absorption in wetted-wall towers and condensation in cooler condensers. A comprehensive survey of current theories and experimental data for both laminar and turbulent films has recently been presented by Fulford (*7, 8*) and the usual survey will not be incorporated here.

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The work described in this paper represents an experimental confirmation of small disturbance theory (*5, 13*), and is therefore somewhat removed from the ultimate goal of understanding the well-developed wavy flow that exists for falling liquid films. Nevertheless, small disturbance theory can now be used with some confidence to predict the effect of surface tension, viscosity, flow rate, surface active agents, etc., on the initial wave structure and should prove useful in assessing trends in the well-developed wavy flow. In addition, the experimental study of small disturbances has led to the detection of two distinct wave velocities, a phenomenon which is easily observed for finite amplitude wave forms in both falling liquid films and two-phase annular flow (*15*).

The flow under consideration is illustrated in Figure 1 which shows the characteristic parabolic velocity profile for steady laminar flow. The velocity profile is given by

$$v_x = u_o \left[1 - \left(\frac{y}{h} \right)^2 \right] \tag{1}$$

where y is taken to be zero at the free surface. The surface velocity is given by

$$u_o = gh^2/2\nu \tag{2}$$

and the film thickness can be expressed in terms of the volumetric flow rate per unit width q as

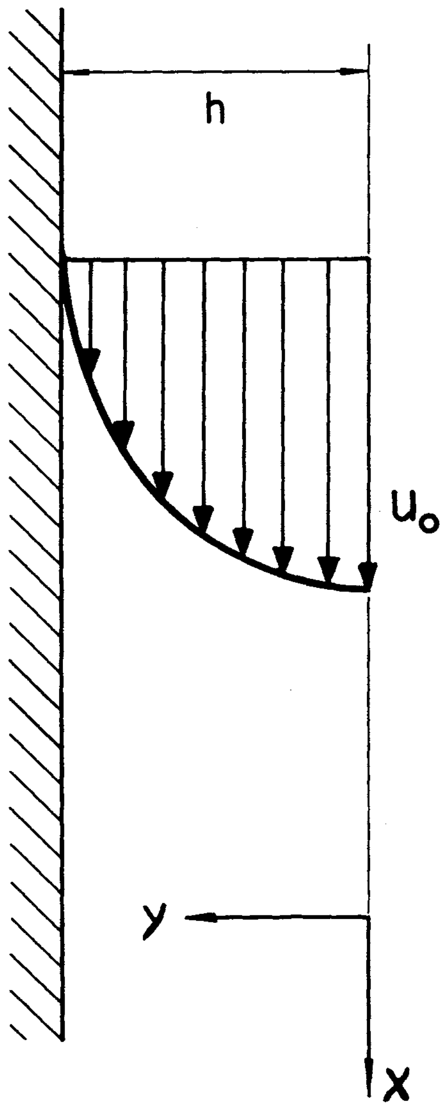

Fig. 1. Undisturbed flow.

$$h = (3\,q\nu/g)^{1/3} \tag{3}$$

The Reynolds number referred to in this paper is defined as

$$N_{Re} = u_o h/\nu \tag{4}$$

Experimental studies lead to the determination of a wave velocity c_r and a wavelength λ. The quantities are generally represented in terms of a dimensionless wave velocity C_r defined by

$$C_r = c_r/u_o \tag{5}$$

and a dimensionless wave number defined by

$$\alpha = 2\pi h/\lambda \tag{6}$$

EXPERIMENTAL METHOD

If the position of the surface, relative to some arbitrary plane, is measured continuously at two neighboring points, the wavelength and wave velocity can readily be determined. There are several ways in which this can be done: Emission from a radioactive tracer can be used as a measure of film depth (*10*) as can light absorption (*9*), or the capacitance between the fluid and a probe may be used to locate the position of the surface (*4, 6*). Wavelength may be determined photographically (*14*) and wave velocity can be determined by means of successive photographs (*3, 4*). This latter technique is only useful if a single wave velocity and wavelength characterize the surface. Use of a radioactive tracer or light absorption (which requires the use of a dye) may lead to adsorption of impurities at the surface and thus alter the flow (*2, 11, 16*). The capacitance probe seems to be the most promising experimental method; however, it requires a sophisticated electronic circuit and suffers from the fact that the probe is not necessarily small compared to the wavelength. Improvements in electronic circuitry will surely eliminate this difficulty in the near future.

The technique used in this work made use of the deflection of a light beam as it passed through the wavy film. The amount of deflection is measured by a photomultiplier tube which, owing to the nonuniformity of the cathode, provides a signal proportional to the deviation of the light beam from the mean position (*18*). The advantage of this scheme is the simplicity of the electronic circuit and optical system. Its disadvantage is that the slope, rather than the position, of the surface is measured. However, the wave velocity and wave length can be obtained as easily from the slope as from the position, so this is not a serious drawback. In addition, this method can be used to measure wave amplitudes by visual observation of the total deflection of the beam. This aspect of the experimental method has not yet been perfected, and only a few preliminary results will be given.

A schematic drawing of the experimental equipment is shown in Figure 2. It consists of a water supply and distributing system for forming the film on the inside of a 1.5 I.D. Truebore Pyrex tube; an optical system for detecting variations in the surface profile at two neighboring points; and a two-channel, strip-chart recorder for monitoring the output of the photomultiplier tubes. Light from a 12 v. incandescent bulb passed through an 18 in. long, 0.5 in. diameter blackened tube to a front surface mirror inclined at 45 deg. This entire assembly was located inside the Truebore pyrex tube. From the mirror, the light passes through two 1/32 in. diameter holes which are ½ in. apart. These small, blackened holes are ¼ in. long and yield two nearly parallel beams of light which pass through the film and glass tubing and strike the cathodes of the two photomultiplier tubes.

An annoying characteristic of photomultiplier tubes is that the cathode is position-sensitive; however, it is just this characteristic which allows us to utilize this rather simple scheme. The method does not require that the cathode sensitivity be a linear function of position, but it is necessary that it be a monotonic function in the region traversed by the oscillating light beam. It was found to be convenient to work in a region where the sensitivity was nearly linear, and such regions could usually be found experimentally for any given tube.

The curves traced on the strip-chart recorder at a Reynolds number of 13 are shown in Figure 3. Channel I represents the signal obtained at 2½ in. from the fluid distributor, while channel II gives the signal at a distance of 3 in. The peaks and troughs of the actual waves give rise to a zero deflection of the light beam; thus these points are represented by some median line (not shown) on the strip-chart curves. The maximum deflection of the light beam occurs when the slope is a maximum; thus the peaks and troughs on the strip-chart curve represent points of maximum slope on the actual waves. If the waves were perfectly uniform, one could not tell which points on the strip-chart curves corresponded to some point on the wave. However, nonuniformities are always present, and it is obvious that points A_1 and A_2 correspond to the same point on some wave.

The wave velocity is readily calculated from the strip-chart by the formula

$$c_r = \frac{\text{(distance between light beams) (velocity of the strip-chart)}}{\text{(distance between points } A_1 \text{ and } A_2\text{)}}$$

Approximately thirty wave velocities were determined for each experimental condition, and the maximum deviation from the average value was always less than $\pm 5\%$. The wavelength is simply the wave velocity divided by wave frequency which is readily obtained from the strip-charts.

In obtaining experimental data care was taken to prevent the water from becoming contaminated with any surface-active material. However, the cleanliness of any water-air interface is always open to question. Before each run the equipment was taken apart, cleaned in a solution of sulfuric acid and potassium chromate, and rinsed in the tap water which was used in these experiments. The water flow was set at a moderate rate and allowed to run for 30 min. before any data were taken. Measurements were always made at night in order to eliminate any extraneous building vibrations.

The distributing system consisted of a constant head reservoir which fed a carefully machined annular flow channel; however, it could not be relied upon to produce a uniform film, and uniformity had to be determined by sight. This was done by adjusting the distributor so that visible waves appeared at a uniform height around the circumference of the tube and the wave velocity was uniform around the tube. This technique is certainly open to criticism and may be one of the main sources of experimental error. However, the practiced eye is quite an able detector and this flaw in the experimental method is not thought to be a serious one.

Measurements were always made as close to the distributor as possible in order to study the region where small disturbance theory should hold. Table 1 lists the wave velocity and wavelength, the Reynolds number, and the distance downstream from the distributor at which the measurement was made. The wave velocity increases slowly with Reynolds number, and the repeated values at any given Reynolds number indicate an experimental error of about $\pm 5\%$. The error in the wave-

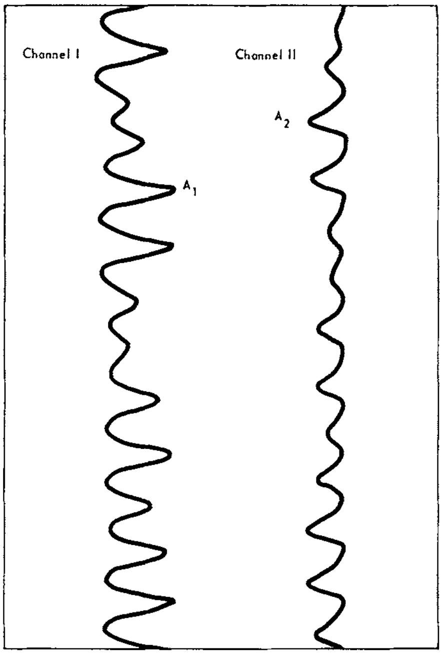

Fig. 3. Strip-chart from recorder.

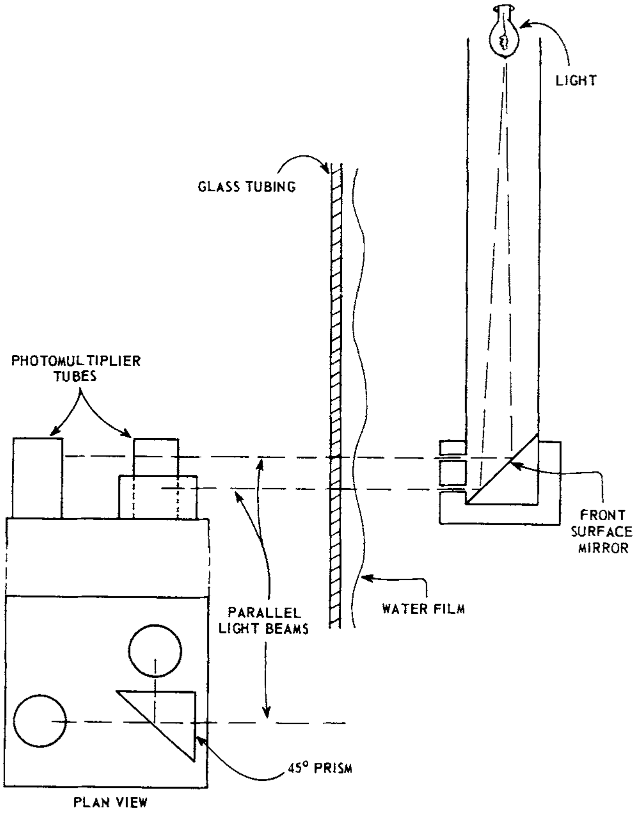

Fig. 2. Experimental system.

TABLE 1. EXPERIMENTAL VALUES OF WAVE VELOCITY AND WAVELENGTH

N_{Re}	c_r, cm./sec.	λ, cm.	Distance from distributor, in.
7.5	12.0	1.05	8
9.5	12.7	0.94	2½
9.5	13.5	1.04	4½
9.5	13.5	0.94	8
13.0	14.4	1.05	2½
13.0	15.1	0.88	2½
13.0	15.4	1.00	4½
13.0	15.6	1.08	8
16.5	17.4	0.87	2½
16.5	17.4	0.86	2½
20.5	19.3	0.69	2½
20.5	19.6	0.90	2½
24.5	20.8	0.81	2½
28.0	22.1	0.77	2½
28.0	21.6	0.77	2½
30.5	23.3	0.82	2½
33.3	24.5	0.87	2½
34.0	23.6	0.82	2½
42.0	25.4	0.90	2½
44.5	24.8	0.78	1
44.5	25.6	0.95	2½
72.5	30.2	0.76	1
72.5	31.5	0.81	1
99.0	36.6	0.93	1
124.0	40.4	1.02	1

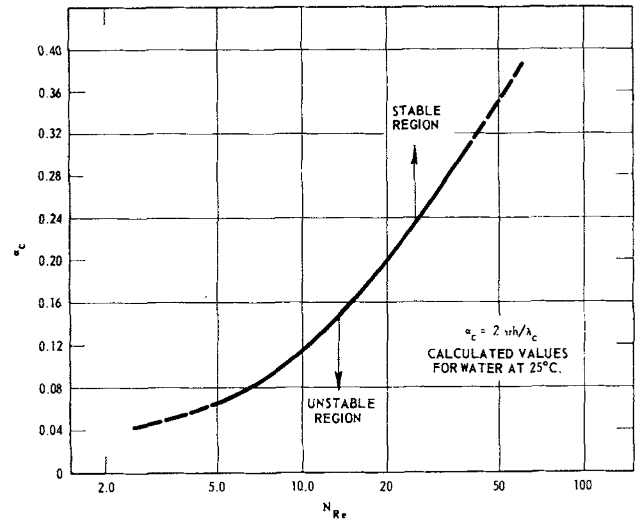

Fig. 4. Critical wave number as a function of Reynolds number.

length is somewhat larger owing to the additional measurement taken from the strip-chart, and is estimated at ± 10%. For practical purposes the wavelength is independent of Reynolds number at a value of 0.9 cm. This is in reasonable agreement with the values obtained by Tailby and Portalski (*14*), who found that λ was essentially constant at a value of 1.0 cm. for Reynolds numbers in the range of 40 to 850.

COMPARISON WITH THEORY

Previous work (*16*) has provided values of α_c, α_m, and C_r as functions of Reynolds number and Weber number, where α_c = critical wave number, that is, the wave number corresponding to a zero growth rate; α_m = wave number for that wave having the maximum growth rate; and C_r = dimensionless wave velocity for that wave having the maximum growth rate.

Figure 4 shows a plot of α_c vs. Reynolds number for water at 25°C., indicating the stable region ($\alpha > \alpha_c$) and the unstable region ($\alpha < \alpha_c$). The assumption always made when comparing experimental data with small disturbance theory is that the infinitesimal wave which has the maximum growth is the wave which grows to finite size and is observed in the laboratory.

The comparison between observed wave numbers and theoretical values of α_m is shown in Figure 5. The agreement is quite good in view of the fact that the experimental values may be in error by ± 10% and the computed values by ± 5% (*16*). The original calculations were limited to Reynolds numbers less than 40, so

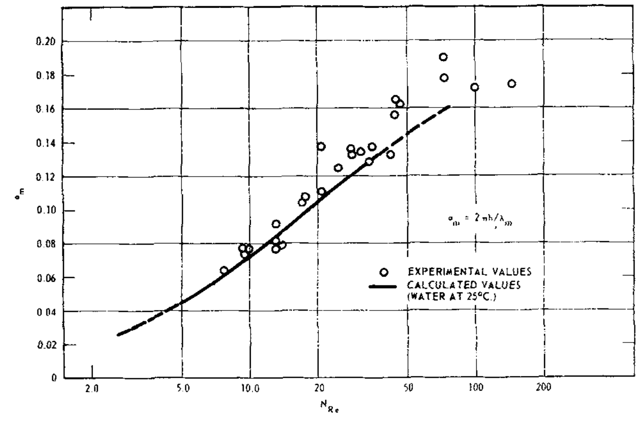

Fig. 5. Comparison between experimental and theoretical wave numbers.

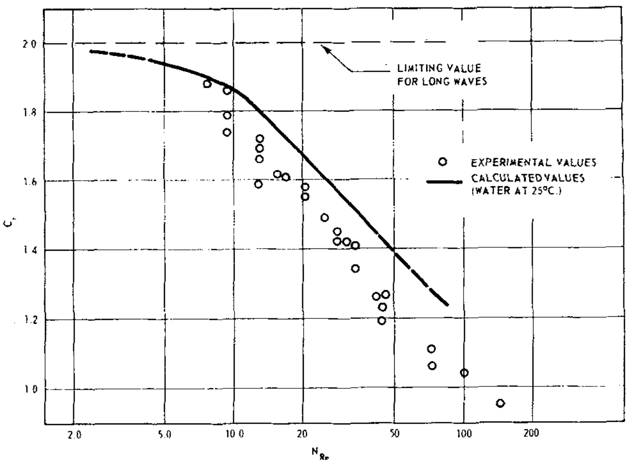

Fig. 6. Comparison between experimental and theoretical wave velocities.

the agreement at higher Reynolds numbers is only inferred by extrapolation. Figure 6 shows the comparison between theory and experiment for the dimensionless wave velocity, and once again the agreement is quite good. The preliminary attempts at measuring the amplitude are listed in Table 2 in terms of the ratio of the amplitude a to the film thickness h as determined by Equation (3). The results certainly indicate the proper trend: a/h increases with both distance and Reynolds number and the order of magnitude for a/h is not unreasonable. The method looks promising. However, much work remains to be done before the growth rates predicted by small disturbance theory can be checked experimentally.

In addition to verifying small disturbance theory and the previous numerical solution of the Orr-Sommerfeld equation, these experiments have lead to the detection of a second wave form. This can be observed at slow strip-chart speeds such as that shown in Figure 7 for a Reynolds number of 9.5. Here the chart speed has been reduced by a factor of ten to 2.5 cm./sec. The secondary wave form is not so obvious as the primary form, and one might be inclined to attribute this to some periodic external disturbance such as building vibration or oscillations in the flow rate. However, visual observations of the waves at a distance of 1 to 2 ft. from the distributor indicated groups of two or three waves which are passed by an occasional high-speed wave. This phenomenon has also been observed in two-phase flows by Taylor, Hewitt, and Lacey (*15*), who designated the two types of waves as ripples and disturbance waves. This situation (if it is a small disturbance phenomenon) would indicate that a second wave form exists with a growth rate comparable to that previous predicted by small disturbance theory. Curves such as those shown in Figure 7 are generated by the presence of two wave trains of equal amplitude but

TABLE 2. AMPLITUDE AS A FUNCTION OF POSITION AND REYNOLDS NUMBER

N_{Re}	a/h	Distance from distributor, in.
13.0	0.07	2½
18.5	0.26	2½
18.5	0.44	3
24.0	0.42	2½
24.0	0.45	3

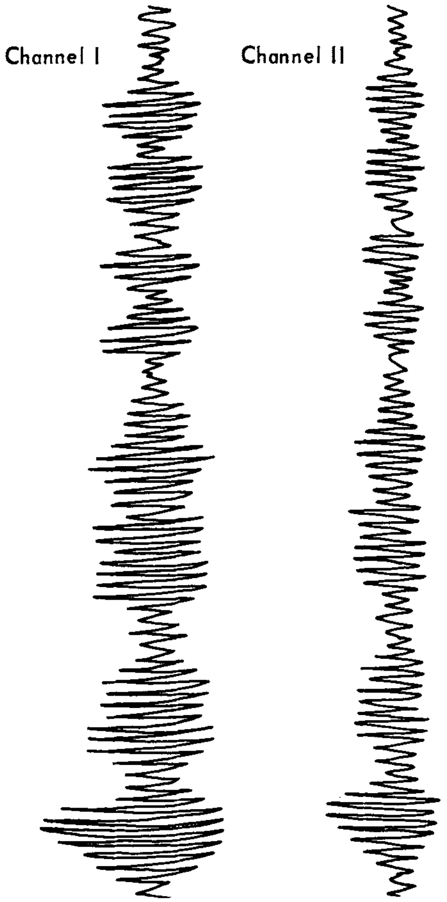

Fig. 7. Low-speed strip-chart curves.

slightly different frequency (*19*). If f_1 and f_2 are the frequencies of the two wave trains, then $(f_1 + f_2)/2$ is the high-frequency component of the curve, and $(f_1 - f_2)/2$ is the low-frequency component of the curve, or the *beat* frequency. The experimental data indicate that

$$f_2 \approx \left(\frac{11}{13}\right) f_1 \tag{8}$$

where f_1 is the primary frequency. For purposes of this discussion we can assume that

$$f_2 \approx f_1 \tag{9}$$

or

$$(\alpha_m C_r)_2 \approx (\alpha_m C_r)_1 \tag{10}$$

Visual observation indicates that $(c_r)_2 > (c_r)_1$ hence

$$(\alpha_m)_2 < (\alpha_m)_1 \tag{11}$$

and the secondary wave has a higher velocity and a longer wavelength.

This would indicate that somewhere between $\alpha = (\alpha_m)_1$ and $\alpha = 0$ there is another maximum in the growth rate–wave number curve. A second maximum would not have been detected by the numerical calculations of Whitaker (*16*), for the computation procedure was stopped after a maximum in the growth rate–wave number curve was reached. However, the analytic solution of Benjamin (*1*) is not subject to this limitation, and Benjamin's results indicate only a single maximum. In addition, a dual maximum has never been observed in other stability analyses, and it does not seem likely that falling liquid films should be a special case. On the basis of the preliminary amplitude measurements, the most satisfactory explanation seems to be that the primary wave form is the result of small disturbances growing to finite disturbances in accordance with theory, while the secondary wave form is related to finite amplitude effects.

CONCLUSIONS

Experimental values for the wave number and wave velocity for falling liquid films have confirmed previous calculations based on small disturbance theory. However, the observed wave form cannot be completely accounted for on the basis of theory and much theoretical and experimental work needs to be done. The experimental technique used in this work should be useful in studying other wave growth processes in the small disturbance region.

NOTATION

a = amplitude, cm.
c_r = wave velocity, cm./sec.
C_r = c_r/u_o, dimensionless wave velocity
g = gravitational acceleration, cm./sec.^{2}
h = undisturbed film thickness, cm.
q = volumetric flow rate per unit width, sq. cm./sec.
u_o = surface velocity, cm./sec.
v_x = velocity in the x direction, cm./sec.
x,y = rectangular Cartesian coordinates, cm.
N_{Re} = $u_o h/\nu$, the Reynolds number

Greek Letters

α = $2\pi h/\lambda$, dimensionless wave number
λ = wavelength, cm.
ν = kinematic viscosity, sq. cm./sec.

Subscripts

c = critical value (zero growth rate)
m = maximum growth rate

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